Exam Advanced Logic

April 9th, 2013

Instructions:

- Put you name and student number on the first page.
- Put your name on subsequent pages as well.
- Do not use pencil (or an erasable pen) or a red pen to make your exam.
- Motivate all your answers.
- If you want to inspect your exam after it has been graded, you can do so by making an appointment with Barteld Kooi.
- Please fill in the anonymous course evaluation.

Good luck!

1. **Induction (10 pt)** Let $\Pi(A)$ be the set of propositional parameters occurring in A. For example if $A = (p \wedge \neg q)$, then $\Pi(A) = \{p, q\}$.

   Now consider the sublanguage $\mathscr{L}_D$ of the language of propositional logic.

   i Each propositional parameter p is a formula of $\mathscr{L}_D$.

   ii If A is a formula of $\mathscr{L}_D$, then so is $\neg A$.

   iii If A and B are formulas of $\mathscr{L}_D$ such that $\Pi(A) \cap \Pi(B) = \emptyset$, then so is $(A \wedge B)$.

   iv Nothing is a formula of $\mathscr{L}_D$ unless it is generated by repeated applications of i, ii and iii.

   Prove by induction that for each formula A of $\mathscr{L}_D$ both A and $\neg A$ are satisfiable. (A formula A is satisfiable iff there exists a valuation $v : P \to \{0, 1\}$ such that $v(A) = 1$.)

2. **Three-valued logics (10 pt)** Show that for all formulas A and B

   $$A \models_{LP} B \text{ iff } \neg B \models_{K_3} \neg A$$

3. **FDE tableau (10 pt)** By constructing a suitable tableau, determine whether the following is valid in **FDE**. If the inference is invalid, provide a counter-model.

   $$\neg p \vee q, \neg(r \wedge s) \vdash (\neg p \vee s) \vee (\neg r \vee q)$$

   NB: Do not forget to draw a conclusion from the tableau.

4. **Fuzzy logic (10 pt)** Determine whether the following holds in $Ł_{\aleph}$ (where $D = \{1\}$). If so, show that if the premises have value 1, so does the conclusion. If not, provide a counter-model.

   $$p \vee q \models q \to (p \to q)$$

5. **Basic modal tableau (10 pt)** By constructing a suitable tableau, determine whether the following is valid in K. If the inference is invalid, provide a counter-model.

   $$\Box\neg p, \Diamond(p \vee \Diamond q) \vdash_K \Diamond\Diamond(q \wedge p)$$

   NB: Do not forget to draw a conclusion from the tableau.

6. **Normal modal tableau (10 pt)** By constructing a suitable tableau, determine whether the following is valid in $K^t_{\tau\phi\beta}$. If the inference is invalid, provide a counter-model.

   $$p \vdash_{K_{\delta\sigma}} \Box\Diamond\Diamond p$$

   NB: Do not forget to draw a conclusion from the tableau.

7. **Soundness and completeness (10pt)** Consider the following branch of a tableau b:

$$\begin{array}{l} \Diamond p, 0 \\ \Box(q \wedge \Diamond p), 0 \\ \neg\Box(p \wedge q), 0 \\ 0r1 \\ p, 1 \\ \Diamond\neg(p \wedge q), 0 \\ 0r2 \\ \neg(p \wedge q), 2 \end{array}$$

Consider the following model $I = \langle W, R, v \rangle$:

$$\begin{array}{lcl} W & = & \{w_1, w_2\} \\ R & = & \{\langle w_1, w_1 \rangle, \langle w_1, w_2 \rangle, \langle w_2, w_1 \rangle\} \\ v_{w_1}(p) & = & 0 \\ v_{w_1}(q) & = & 0 \\ v_{w_2}(p) & = & 1 \\ v_{w_2}(q) & = & 1 \end{array}$$

Show that I is faithful to b. (This means you have to provide a function $f : \mathbb{N} \to W$ and show that it has the desired properties.)

8. **First-order modal tableau (10 pt)** By constructing a suitable tableau, determine whether the following is valid in CK. If the inference is invalid, provide a counter-model.

$$\Box\neg\exists x(Ax \wedge Bx), \exists x\Diamond(Cx \wedge Ax) \vdash_{CK} \Diamond\exists x(Cx \wedge \neg Bx)$$

NB: Do not forget to draw a conclusion from the tableau.

9. **Default logic (10 pt)** Consider the folowing set of default rules:

$$D = \{d_1 = \frac{p : q}{q}, \qquad d_2 = \frac{p : q \wedge r}{s \wedge r}, \qquad d_3 = \frac{q : \neg r}{s \wedge \neg r}\},$$

and initial set of facts:

$$W = \{p\}.$$

Recall that a formula φ is a *skeptical consequence* of (W, D) if and only if φ is true in every extension of (W, D), while it is a *credulous consequence (goedgelovig gevolg)* of (W, D) if and only if φ is true in some extension of (W, D).

(a) Draw the process tree of this default theory.

(b) Is $\neg r$ a skeptical consequence of this theory?

(c) Is $q \wedge \neg r$ a credulous consequence of this theory?

1 Base case: Let A be a propositional variable p.
Let v_1 be a valuation such that $v_1(p)=1$.
Let v_2 be a valuation such that $v_2(p)=0$ (and so $v_2(\neg p)=1$)
It follows that both p and $\neg p$ are satisfiable.

Inductive hypothesis. Let A and B be arbitrary formulas of L_P.
Suppose that A and $\neg A$ are satisfiable.
Suppose that B and $\neg B$ are satisfiable as well.

Induction step. Suppose that $\Pi(A) \cap \Pi(B) = \emptyset$

negation. Consider $\neg A$. By the inductive hypothesis it follows that $\neg A$ is satisfiable. Since A is satisfiable as well $\neg\neg A$ must also be satisfiable.

conjunction. Consider $A \wedge B$. From the inductive hypothesis we know that there are valuations v_1 and v_2 such that $v_1(A)=1$ and $v_2(B)=1$. Now define v_3 such that

$$v_3(p) = \begin{cases} v_1(\cancel{A}p) & \text{if } p \in \Pi(A) \\ v_2(p) & \text{otherwise} \end{cases}$$

Clearly $v_3(A \wedge B)=1$

Consider $\neg(A \wedge B)$. From the inductive hypothesis it follows that there is a valuation v_4 such that $v_4(\neg A)=1$.
Therefore $v_4(\neg(A \wedge B))=1$
So both $A \wedge B$ and $\neg(A \wedge B)$ are satisfiable

Conclusion. By induction we conclude that all formulas and their negations in L_P are satisfiable.

2. $\Rightarrow$ Suppose that $A \models_{RM_3} B$. Therefore for all valuations v it holds that if $v(A) \in \{i,1\}$, then $v(B) \in \{i,1\}$. Suppose that $v(\neg B) = 1$, therefore $v(B) = 0$, so $v(B) \notin \{i,1\}$. Hence $v(A) \notin \{i,1\}$ and so $v(A) = 0$ and $v(\neg A) = 1$. Therefore $\neg B \models_{K_3} \neg A$.

$\Leftarrow$ Suppose that $\neg B \models_{K_3} \neg A$. Therefore for all valuations v it holds that if $v(\neg B) = 1$, then $v(\neg A) = 1$. In other words if $v(B) = 0$, then $v(A) = 0$. Suppose that $v(A) \in \{i,1\}$, therefore $v(A) \neq 0$. Hence $v(B) \neq 0$, and so $v(B) \in \{i,1\}$. Therefore $A \models_{RM_3} B$.

3

$\neg p \vee q, +$

$\neg(r \wedge s), +$

$(\neg p \vee s) \vee (\neg r \vee q), -$

$\neg p \vee s, -$

$\neg r \vee q, -$

$\neg p, -$

$s, -$

$\neg r, -$

$q, -$

$\neg p, +$	$q, +$
$\times$	$\times$

The tableau closes, so $\neg p \vee q, \neg(r \wedge s) \vdash (\neg p \vee s) \vee (\neg r \vee q)$

4. Suppose that $v(p \vee q) = 1$. Since $v(p \vee q) = \max(v(p), v(q))$ it follows that $v(p) = 1$ or $v(q) = 1$.

Suppose that $v(q) = 1$. Therefore $v(p) \leq v(q)$. Therefore $v(p) \ominus v(q) = 1$, and so $v(p \to q) = 1$, and also $v(q \to (p \to q)) = 1$

Suppose that $v(p) = 1$, and without loss of generality that $v(q) < 1$. Therefore $v(p) > v(q)$ and $v(p) \ominus v(q) = 1 - (v(p) - v(q)) = v(q)$. Since $v(q) \leq v(q)$, $v(q) \leq 1 - (v(p) - v(q))$, so $v(q \to (p \to q)) = 1$

In both cases $v(q \to (p \to q)) = 1$. So $p \vee q \vDash q \to (p \to q)$

5. $\Box\neg p, 0$

$\Diamond(p \vee \Diamond q), 0$

$\neg\Diamond\Diamond(q \wedge p), 0$

$\Box\neg\Diamond(q \wedge p), 0$

$0r1$

$p \vee \Diamond q, 1$

$\neg p, 1$

$\neg\Diamond(q \wedge p), 1$

$\Box\neg(q \wedge p), 1$

$p, 1$	$\Diamond q, 1$
$\times$	$1r2$
	$q, 2$
	$\neg(q \wedge p), 2$

$\neg q, 2$	$\neg p, 2$
$\times$	$\uparrow$
	open and complete

Counter model

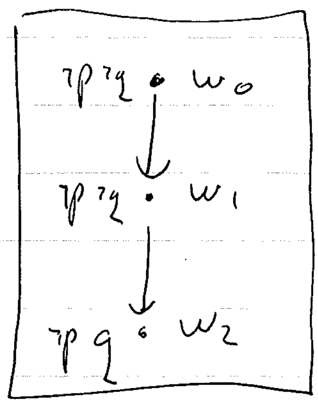
$W = \{w_0, w_1, w_2\}$

$R = \{\langle 0,1\rangle, \langle 1,2\rangle\}$

$V_0(p) = 0 \qquad V_0(q) = 0$

$V_1(p) = 0 \qquad V_1(q) = 0$

$V_2(p) = 0 \qquad V_2(q) = 1$



6.

$p, 0$

$\neg\Box\Diamond\Diamond p, 0$

$\Diamond\neg\Diamond\Diamond p, 0$

$0r1$

$\neg\Diamond\Diamond p, 1$

$\Box\neg\Diamond p, 1$

$0r2$

$2r1$

$2r0$

$1r2$

$\neg\Diamond p, 2$

$\Box\neg p, 2$

$\neg p, 0$

$\times$

The tableau closes, so $p \vdash_{K\delta\sigma} \Box\Diamond\Diamond p$

7 Take $f(0) = w_1$
$f(1) = w_2$
$f(2) = w_1$

✓ $v_{w_1}(\Diamond p) = 1$
✓ $v_{w_1}(\Box(q \wedge \Diamond p)) = 1$
✓ $v_{w_1}(\neg \Box(p \wedge q)) = 1$
✓ $v_{w_2}(p) = 1$
✓ $v_{w_1}(\Diamond \neg(p \wedge q)) = 1$
✓ $v_{w_1}(\neg(p \wedge q)) = 1$

✓ $w_1 R w_2$
✓ $w_1 R w_2$

8

$$\Box\neg\exists x(Ax\wedge Bx),0$$
$$\exists x\Diamond(Cx\wedge Ax),0$$
$$\neg\Diamond\exists x(Cx\wedge\neg Bx),0$$
$$\Box\neg\exists x(Cx\wedge\neg Bx),0$$
$$\Diamond(Cd\wedge Ad),0$$
$$0r1$$
$$Cd\wedge Ad,1$$
$$Cd,1$$
$$Ad,1$$
$$\neg\exists x(Ax\wedge Bx),1$$
$$\neg\exists x(Cx\wedge\neg Bx),1$$
$$\forall x\neg(Ax\wedge Bx),1$$
$$\forall x\neg(Cx\wedge\neg Bx),1$$
$$\neg(Ad\wedge Bd),1$$

$$\neg(Cd\wedge\neg Bd),1$$

$\neg Ad,1$
$\times$

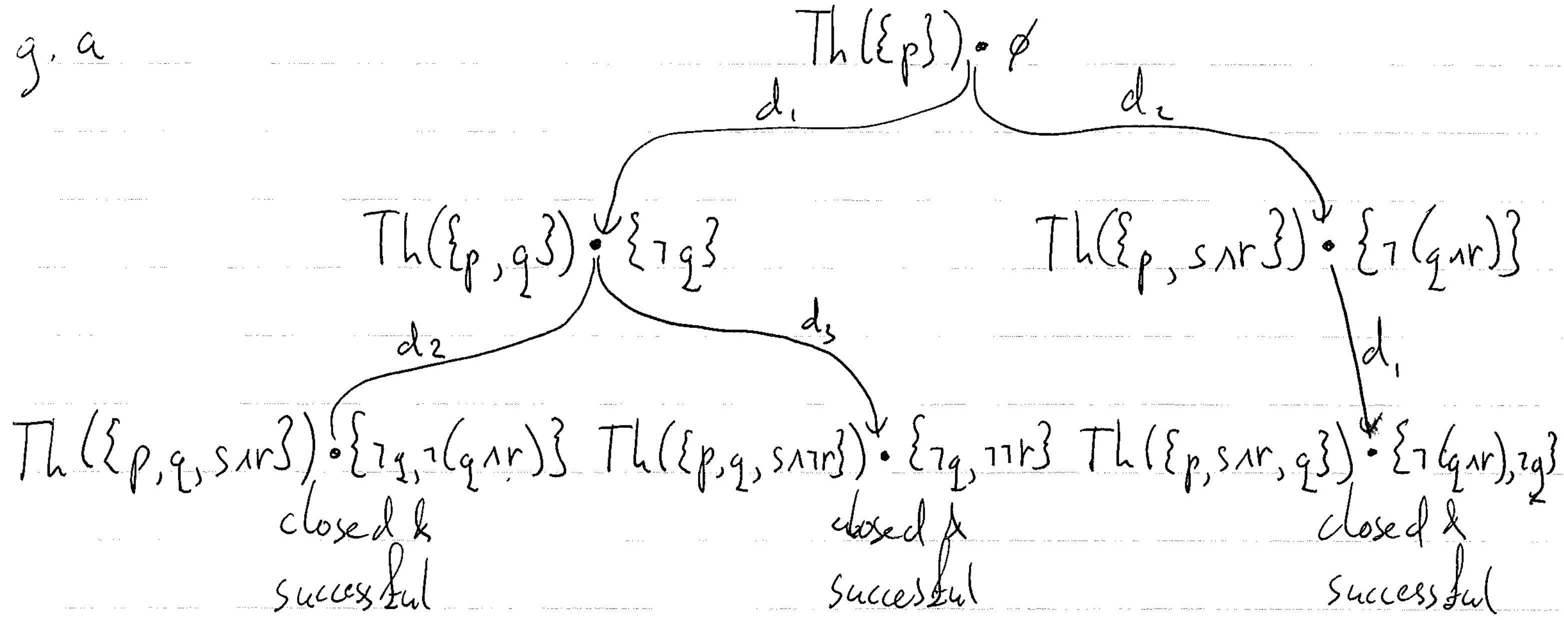
$\neg Bd,1$

$\neg Cd,1$
$\times$

$\neg\neg Bd,1$
$\times$

The tableau closes, so $\Box\neg\exists x(Ax\wedge Bx),\exists x\Diamond(Cx\wedge Ax)\vdash_{CK}\Diamond\exists x(Cx\wedge\neg Bx)$

9. a



b. No, $\neg r$ is not a skeptic consequence of the theory. It is in some extensions, but not all

c. Yes, $q \wedge \neg r$ is a credulous consequence of the theory. It is in one extension.